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Finite width effects and gauge cancellations in W- and Z-boson production in the framework of modified perturbation theory

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Abstract. The processes of production and subsequent decay of W- and Z-bosons in e^+e^- collisions are considered in a recently proposed modified perturbation theory (PT), based on a direct expansion of probabilities instead of amplitudes. In such an approach the non-integrable singularities in the phase space, which are intrinsic in the conventional PT, appear as singularities in the coupling constant (with subsequent compensation by the decay factors of unstable particles). In the present paper the systematic investigation of the modified PT is carried out. The results are compared with the results of the conventional approach, based on the calculation of the amplitude with Dyson resummation. A solution to the problem of the loss of one-loop PT order in the resonance region is found. On the basis of this solution the proof of gauge cancellations in any order of the modified PT is given. A simple generalization of the fermion-loop scheme is proposed which provides a complete description of W-pair production in a next-to-leading order approximation.

1 Introduction

In many applications of the standard model connected with the present and future collider experiments, one should take into account the effects of the instability of W- and Z-bosons (as well as of Higgs boson, top quark etc.) [1]. In quantum field theory the conventional way to take into account an instability consists in the Dyson resummation of the self-energies of unstable particles [2]. This procedure avoids non-integrable phase-space singularities caused by the processes of production and decay of intermediate unstable particles. However, Dyson resummation leads to a deviation from the scheme of fixed-order calculations in the framework of perturbation theory (PT). In gauge theories this results in the violation of the Ward identities (WI) and loss of gauge invariance [3–5]. This fact leads to loss of the control of the high-energy behavior of the theory and to the emergence of large errors in the description of particular processes.

In the case of single Z-boson production (LEP1) and within the precision defined by one-loop corrections to the vertex functions, the problem of the gauge invariance was solved ad hoc (in fact, the mentioned precision implies the approximation of next-to-leading order, NLO). The most consistent scheme of calculations was described in [5], where only the gauge-invariant contributions to the self-energy were Dyson resummed, while the gauge-dependent ones were considered by the conventional PT. As a result, the amplitude could be presented as a product of gauge-independent factors, two vertex and one resonant. Nevertheless, this result is not universal. Anyway, now it is not clear whether this holds within the next order of precision determined by two-loop corrections to the vertex functions.

In the case of pair production of unstable particles (LEP2) the amplitude cannot be presented in the framework of Dyson resummation in a completely gauge-invariant form[1]. The hopes, nevertheless, for any further progress in such calculations were usually connected with the rather general idea of the determination of the minimal set of Feynman diagrams that are necessary for compensating the gauge violation by the Dyson resummed self-energies. For this purpose within the NLO approximation the fermion-loop scheme was proposed [8,9]. Nevertheless, the bosonic corrections were not taken into consideration in this approach. In order to solve this problem a generalization of the fermion-loop scheme was proposed [10], defined in terms of the formalism of the background-field method. It solved the problem of the bosonic corrections and, moreover, it remained in force also beyond the one-loop approximation. However, this approach could not solve another problem, which was in the fermion-loop scheme, too. This problem concerns the incompleteness of the description within the declared precision of description because of the loss of one-loop order of PT in the resonance region [11,12].

[1] It should be noted that there is an alternative approach called the pole scheme, described in [6] and then elaborated in numerous papers. The gauge invariance in this scheme is initially maintained, but, unfortunately, an algorithm for the evaluation of corrections is not developed completely (see [7] and the discussion in Sect. 8)

Let us consider in more detail the latter phenomenon. The point is that the denominator of the unstable-particle propagator in the resonance region, $p^2 - M^2 = O(g^2)$, is of order $O(g^2)$, but not $O(1)$. So in this region the Dyson resummed one-loop self-energy actually makes a contribution in the leading order, but not in NLO. Therefore, in order to complete the NLO approximation the two-loop correction to the self-energy should be Dyson resummed.

However, the Dyson resummation of the two-loop correction, made without taking into consideration the two-loop corrections to the vertex functions, hardly leaves a chance to maintain WI. Consideration of all two-loop corrections maybe solves the problem. Nevertheless, this solution is certainly impractical [11,12]. Besides, the question remains unclear how the two-loop corrections to the vertex functions, which in fact contribute in NNLO, can compensate the gauge violation that occurs in NLO. So anyway the problem of the loss of one-loop PT order has to be solved in a practical fashion.

The present paper proposes the solution to this problem in the framework of the modified PT [13]. Its basic idea is the expansion of direct probabilities in powers of the coupling constant instead of amplitudes. (The amplitudes prior to the calculation of the probabilities are considered to be full.) Such an order of operations, taken together with ideas of the theory of generalized functions [14,15], allows one to trace the fundamental connection between the origin of the phase-space non-integrable singularity and the loss of one-loop PT order in the resonance region. (In fact this loss manifests itself as the emergence of an extra singularity in the coupling constant instead of the phase-space singularity.) Moreover, while considering the probability the contribution of the Dyson resummed two-loop correction (in the amplitude) may be reproduced within the given precision in the form of an additive anomalous term. Owing to the additivity of this term, it becomes possible to give an independent proof of the gauge cancellations in the probability. It is worth mentioning that this result does not mean that the inclusion of two-loop corrections only to denominators of “unstable” propagators (without the inclusion to vertices) does not lead to violation of WI in the amplitude. This means that contributions that violate WI in the amplitude turn out to be beyond the given precision in the probability. By its nature this phenomenon is connected with the effect of changing of the order of individual contributions in the probability due to the emergence of a singularity in the coupling constant instead of the phase-space singularities.

Below we elaborate the above-stated ideas in any order of the modified PT and give a general proof of the gauge cancellations in the probability within the given precision. Notice that the latter outcome was practically anticipated in the pioneering work of [13], presenting the modified PT. However, the reasoning of this work in the part that concerns the gauge cancellations was not complete. Indeed, it was based on a comparison with results of the conventional approach, but it overlooked the problem of losing one-loop PT order in the resonance region. Besides, it omitted the problem of the difference between remainders of the expansions of the amplitude and the corresponding probability. Nevertheless, proceeding from only the amplitude it is impossible to estimate in a mathematically correct way the remainder of the expansion of the probability, since in the resonance region the expansion of amplitude faces the ambiguity of the expression $0/0$.

In order to investigate in detail the above-mentioned problem we first perform a systematic study of the modified PT approach. Then we proceed by applying the results in the theory of electroweak interactions. In the general case we find a practical way to keep a fixed precision without violating gauge cancellations in the framework of the background-field method. Within NLO we also find the solution in the usual formalism, basing ourselves on the results of the fermion-loop scheme.

This paper is organized as follows. In Sect. 2 a statement of the problem is discussed by a simplified example of a single unstable particle production. The basic formulas of the modified PT are derived in Sect. 3 (as a whole, the content of this section follows [13]). The properties of the expansion of the unstable propagator squared are studied in Sect. 4. Section 5 discusses briefly the soft-photon problem. In Sect. 6 the general proof is given of the gauge cancellations in processes mediated by unstable particle production. Section 7 is devoted to the construction of the generalization of the fermion-loop scheme in NLO approximation. In Sect. 8 the results are discussed.

2 Unstable propagator in AO: statement of the problem

In this section we consider the structure of the denominator of an unstable propagator. We reject, for a moment, all factors in the numerator. Then let us consider the propagator in the following form:

$$\Delta(\alpha;\tau) = \frac{1}{M^2 - p^2 - \Sigma} = \frac{1}{\tau - \alpha h(\tau) - \mathrm{i}\alpha f(\tau)}. \qquad (1)$$

Here $\alpha = g^2/(4\pi)$ is the coupling constant squared, $\tau = M^2 - p^2$ is the kinematic variable, M and p are the mass and momentum of the unstable particle (the mass squared is considered without the conventional $-\mathrm{i}0$), Σ is the renormalized self-energy[2], αh and αf are its real and imaginary parts with the extracted factor α. Here we do not fix the scheme of UV renormalization because the results will not depend on it (see below). By definition, the property of instability means that $f \neq 0$ in some neighborhood of the point $\tau = 0$. Owing to causality we assume that $f > 0$ in this neighborhood. In what follows we assume that the size of this neighborhood is of the order of $O(\alpha)$ and that it contains a solution to the equation $\tau - \alpha h(\tau) = 0$. Generally speaking, the function $h(\tau)$ may be non-zero at $\tau = 0$.

The probability of the process of the production and decay of an unstable particle is defined by an integral over

[2] In the case of vector bosons the self-energy includes two structures, $g_{\mu\nu}$ and $p_\mu p_\nu$. Formula (1) represents the contribution of the first structure only. The effect of the presence of the $p_\mu p_\nu$-structure will be discussed later on

some region of the kinematic variable of the propagator squared $\mathcal{W}(\alpha;\tau)$, multiplied by some weight function,

$$P(\alpha) = \int \mathrm{d}\tau \; \varphi(\tau) \; \mathcal{W}(\alpha;\tau), \tag{2}$$

$$\mathcal{W}(\alpha;\tau) \equiv |\Delta(\tau)|^2 = \frac{1}{[\tau - \alpha h(\tau)]^2 + \alpha^2 f^2(\tau)}. \tag{3}$$

The weight function $\varphi(\tau)$ stands for the complementary part of the unitarity diagram with respect to the given propagator squared. It includes the photon radiation from the initial and final charged states. In the general case $\varphi(\tau)$ includes also the hardware factors of experimental devices.

By virtue of the property of $f \neq 0$ the function $\mathcal{W}(\alpha;\tau)$ is finite and, therefore, is integrable in a neighborhood of $\tau = 0$. However, in the limit $\alpha \to 0$ the non-integrable singularity $1/\tau^2$ appears in $\mathcal{W}(\alpha;\tau)$. This fact means the impossibility of direct application of the conventional PT in the case of processes mediated by unstable particles. Nevertheless, the expansion in the coupling constant does exist in the probability $P(\alpha)$.

Actually, the origin of non-integrable singularity means that the *result of integration* of $\mathcal{W}(\alpha;\tau)$ with a weight function $\varphi(\tau)$ includes a singularity in α at $\alpha \to 0$. If one extracts this singularity, then the expansion of the remaining integral becomes possible. In the case of a power-like singularity, this expansion will be a Laurent expansion, but not a Taylor one. (Let us note that the weight function $\varphi(\tau)$ actually depends on the parameter α, including it, in particular, as a factor. Therefore, the expansion of the integral ultimately may take a Taylor form. However, a priori the kind of the singularity is not known. So in order to study the problem, we first consider $\varphi(\tau)$ as a function independent from α. We also assume that $\varphi(\tau)$ is a rather smooth function and generally is non-zero at $\tau = 0$.)

Let us study the kind of singularity of the integral. Since the singularity originates from integration over small τ, we may keep in the functions $h(\tau)$ and $f(\tau)$ only their leading terms of the asymptotic expansion at $\tau \to 0$, i.e. approximate them by $h_0 = h(0)$ and $f_0 = f(0)$. As a result, up to inessential corrections (which will be calculated later on), we get the approximation as follows:

$$\mathcal{W}(\alpha;\tau) \cong \frac{1}{[\tau - \alpha h_0]^2 + \alpha^2 f_0^2}. \tag{4}$$

By virtue of the homogeneity one can deduce from (4) that the integral of $\mathcal{W}(\alpha;\tau)$ with the weight function $\varphi(\tau)$ has a singularity of $1/\alpha$. Indeed, let us divide the range of integration into $|\tau| > \text{const} \times \alpha$ and $|\tau| < \text{const} \times \alpha$ with large enough "const". The integration over the first range gives a finite contribution, while the integration over the second range gives the above-mentioned singularity (this may be verified by a change of the integration variable). Moreover, the coefficient at the singularity is proportional to f_0^{-1} and does not depend on h_0. In fact, f_0 may be included into the normalization of α, whereas h_0 does not make a contribution in the leading order. Indeed, setting $h_0 = 0$ does not lead to a singularity in τ. For similar reasons the weight function $\varphi(\tau)$ gives a contribution to the leading-order term as a trivial factor $\varphi(0)$.

So, despite the fact that the expansion of the integrand is an incorrect operation, the expansion of the *result of integration* is worthwhile. Moreover, some properties of such an expansion may be determined before the actual calculation of the integral. For a systematic determination of the properties there is a special method called the asymptotic operation, AO [16–18]. Its key point is the transition to an extended interpretation of the integrand as a product of a kernel of a *generalized* function on a *test* function [14, 15]. In fact this means that the integral is interpreted as a continuous linear functional on a test function. When the integral is well defined (*before* expanding the integrand) the above-mentioned generalization does not lead to any modification. However, after the formal expansion of the integrand the new interpretation allows one to give meaning to the non-integrable terms of the expansion.

Thus, the problem of the expansion of the integrand may be solved basically within the method of generalized functions. Next, it is necessary to consider the asymptotic properties of the expansion. For this purpose an ambiguity of the extension of the interpretation of non-integrable functions in the integrand is used. Generally, it is well known (e.g. from experience of the UV renormalizations) that elimination of divergences may be accompanied by ambiguities. When an integral is determined by the method of generalized functions, the ambiguities are described by means of so-called counterterms, which are proportional to the delta-function or its derivatives located strictly in the point of the non-integrable singularity[3]. In the AO framework the coefficients at counterterms are unambiguously fixed by the requirement of the reconstruction of the result which should be obtained from an expansion of the initial integral. (Let us recall that before expansion of the integrand the integral was well defined and there were no ambiguities in it.) Moreover, AO gives a practical recipe of the calculation of these coefficients in each order of the expansion prior to a calculation of the integral. The resulting counterterms contain complete information about the terms which are singular in the parameter of the expansion. Simultaneously the counterterms may also contain some non-singular contributions which correct the asymptotic property of the expansion.

In the above example the counterterm that describes the leading term of the asymptotic expansion of $\mathcal{W}(\alpha;\tau)$ is $c/(\alpha f_0) \times \delta(\tau)$ where c is some numerical factor. In the given case the value of c, as well as the appearance of this counterterm, follows from the formula which is well known in the theory of generalized functions,

$$\lim_{\alpha \to 0} \frac{\alpha}{\tau^2 + \alpha^2} = \pi\delta(\tau). \tag{5}$$

[3] Let us emphasize that the introduction of counterterms is a general place in the theory of generalized functions (see e.g. [15]). Actually this idea was used by Bogoliubov [19,20] for establishing the R-operation. In the AO context the term "counterterms" was introduced [16,17] in order to emphasize the analogy with the theory of UV renormalizations

Thanks to this formula the expansion of $\mathcal{W}(\alpha;\tau)$ up to an $O(1)$ correction is defined unambiguously and turns out to be the delta-function only. Up to the $O(\alpha)$ correction the expansion is non-trivial. Its most general form is

$$\begin{aligned}\mathcal{W}(\alpha;\tau) &= \pi/(\alpha f_0)\delta(\tau) \\ &\quad + \left[1/\tau^2\right] + c_0\delta(\tau) + c_1(-)\delta'(\tau) + O(\alpha).\end{aligned} \tag{6}$$

Here $\left[1/\tau^2\right]$ is the generalized function defined with some prescription. (The most common but not mandatory prescription is the principal value. Later on we omit the square brackets.) The last two terms in (6) are counterterms that correct the contributions of the first two terms.

In the general case the complete determination of the generalized function $1/\tau^2$ may be done in the following way [15]. First, one makes two subtractions in the test function $\varphi(\tau)$ in some neighborhood of $\tau = 0$ by replacing $\varphi(\tau)$ to $\varphi(\tau) - \varphi(0) - \tau\varphi'(0)$. As a result, the non-integrable singularity in $1/\tau^2$ becomes compensated. (In fact this is one of possible prescriptions.) Then in order to describe the ambiguity emerging with this subtraction, one has to add two counterterms to $1/\tau^2$. One counterterm is proportional to the delta-function and the other one is proportional to its first derivative (both counterterms correspond to the above subtractions). The coefficients at the counterterms must be determined in such a way [16–18] as to guarantee the asymptotic properties of the expansion of the integral at order $O(1)$. Their values depend on h and f, and on the choice of the prescription in $1/\tau^2$, but the sum of all terms will not depend on the prescription.

The above process may be continued. The next term of the formal expansion of $\mathcal{W}(\alpha;\tau)$ is $2\alpha h(\tau) \times 1/\tau^3$. For its complete determination one needs three counterterms which include the delta-function, its first and its second derivatives. The coefficients at counterterms are fixed by the requirement of the asymptotic properties of the expansion. The practical recipe of the calculation of the coefficients is presented in the next section.

3 Calculation of counterterms

Let us show the technique of the calculation of counterterms on the example of the AO expansion of $\mathcal{W}(\alpha;\tau)$ up to $O(\alpha^2)$. Since the leading term of this expansion is of order $O(\alpha^{-1})$, the mentioned precision is sufficient for the determination of the next-to-next-to-leading order (NNLO) approximation. Such a precision seems to be sufficient for most of the practical applications. It is sufficient also for understanding the properties of the AO expansion of $\mathcal{W}(\alpha;\tau)$.

The general structure of the AO expansion of $\mathcal{W}(\alpha;\tau)$ within the considered precision is as follows:

$$\mathcal{W}(\alpha;\tau) = \frac{1}{\tau^2} + \frac{2h(\tau)}{\tau^3}\alpha + E(\tau) + O(\alpha^2). \tag{7}$$

Here the first two terms represent the result of the formal expansion of $\mathcal{W}(\alpha;\tau)$. For definiteness, we treat the poles with respect to τ in the sense of principal value. Below we recall two equivalent definitions of the principal value:

$$\begin{aligned}VP\frac{1}{\tau^n} &= \frac{1}{2}\left[\frac{1}{(\tau+\mathrm{i}0)^n} + \frac{1}{(\tau-\mathrm{i}0)^n}\right] \\ &= \frac{(-)^{n-1}}{(n-1)!}\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}\ln|\tau|.\end{aligned} \tag{8}$$

The derivatives in the last expression are understood in the sense of generalized functions, i.e. they must be switched to the test function via formal integration by parts.

The quantity $E(\tau)$ in (7) represents the sum of counterterms. In this case these are proportional to the delta-function, its first and its second derivatives:

$$E(\tau) = \sum_{n=0}^{2} \frac{(-)^n c_n}{n!}\delta^{(n)}(\tau). \tag{9}$$

In what follows we assume that $h(\tau)$ and $f(\tau)$ together with their second derivatives are regular functions in some neighborhood of $\tau = 0$[4]. At first, for simplicity we suppose that the functions h and f include one-loop contributions only. A generalization to the multi-loop case is considered below in this section.

We wish to define a procedure for the determination of the coefficients c_n, $n = 0,1,2$, for an arbitrary $\varphi(\tau)$ function, which decreases rapidly enough at infinity. This procedure should provide the following:

$$\int_{-\infty}^{+\infty} \mathrm{d}\tau\varphi(\tau)\left[\mathcal{W}(\alpha;\tau) - \frac{1}{\tau^2} - \frac{2h(\tau)}{\tau^3}\alpha - E(\tau)\right] = O(\alpha^2). \tag{10}$$

Note, that for the solution of this problem we do not have to know all information about the function $h(\tau)$ in the third term in square brackets. We need only have knowledge about three terms of its asymptotic expansion,

$$h(\tau) = h_0 + \tau h_0' + (\tau^2/2)h_0'' + o(\tau^2). \tag{11}$$

This property follows from the fact that the remainder $o(\tau^2)$ cancels the non-integrable singularity $1/\tau^3$ in (10).

Now let us substitute (9) into (10) and, following [16], present the test function $\varphi(\tau)$ as a linear combination of three basis functions $\varphi_n(\tau)$, $n = 0,1,2$, such that $\varphi_n^{(k)}(0) = \delta_n^k$, where δ_n^k is the Kronecker symbol. As a result we get

$$c_n = \int_{-\infty}^{+\infty} \mathrm{d}\tau\varphi_n(\tau)\left[\mathcal{W}(\alpha;\tau) - \frac{1}{\tau^2} - \frac{2h(\tau)}{\tau^3}\alpha\right] + O(\alpha^2). \tag{12}$$

[4] If the unstable particle is able to interact with massless particles (photons), this property does not occur. Nevertheless, supplying the photons with mass we restore the analyticity inside a neighborhood defined by the generated mass gap. This is enough for our purposes (see discussion in Sect. 5)

It is obvious from (12) that within the given precision the coefficients c_n do not depend on the choice of test functions. Indeed, any other test functions $\widetilde{\varphi}_n(\tau)$ possessing the same property, $\widetilde{\varphi}_n^{(k)}(0) = \delta_n^k$, lead to the coefficients $\widetilde{c}_n$ instead of c_n. However, the difference $\widetilde{c}_n - c_n$ amounts to a quantity of order $O(\alpha^2)$, because the difference between the corresponding integrals is controlled by the test function $\widetilde{\varphi}_n(\tau) - \varphi_n(\tau)$, which equals zero at $\tau = 0$ together with its first and second derivatives. It is easy to see that the integral in formula (12) with such a weight function is of order $O(\alpha^2)$. Due to this property, without loss of generality one may choose the test functions to be the step-like ones. Namely, we may set $\varphi_n(\tau) = \tau^n \times \theta(|\tau| < \Lambda)$. Then, with $n = 0, 1, 2$, we have

$$c_n = \int\limits_{-\Lambda}^{+\Lambda} \mathrm{d}\tau \tau^n \left[\mathcal{W}(\alpha;\tau) - \frac{1}{\tau^2} - \frac{2h(\tau)}{\tau^3}\alpha \right] + O(\alpha^2). \quad (13)$$

Formula (13) basically solves the problem stated above. However, it is still too complicated for practical usage since, generally speaking, through $\mathcal{W}(\alpha;\tau)$ it contains the dependence on the unknown functions $h(\tau)$ and $f(\tau)$. Moreover, (13) contains a lot of superfluous information because the integral on the r.h.s. includes contributions beyond the given precision. In particular, the dependence on the cutoff parameter Λ is of this kind.

The problem may be solved by a homogenization procedure [18], i.e. by specific transformations in the integrand. Namely, let us do the substitutions $\tau \to \xi\tau$, $\alpha \to \xi\alpha$. Then the result we expand in powers of ξ, and in the end we set $\xi = 1$. Each term of this (secondary) expansion is proved to be a homogeneous function of τ and α. So it leads to a strictly definite contribution in the power in α to the integral, which now may be considered without the cutoff. The first term $c_n^{(0)}$ of this expansion gives the leading-order contribution to the coefficient c_n,

$$c_n^{(0)} = \int\limits_{-\infty}^{+\infty} \mathrm{d}\tau \tau^n \left[\frac{1}{(\tau - \alpha h_0)^2 + \alpha^2 f_0^2} - \frac{1}{\tau^2} - \frac{2h_0}{\tau^3}\alpha \right]. \quad (14)$$

The next term of the expansion of homogenization gives the correction term $c_n^{(1)}$, etc. On counting of powers we have $c_n^{(0)} \sim \alpha^{n-1}$, $c_n^{(1)} \sim \alpha^n$, etc. Adding the necessary number of $c_n^{(i)}$ we get c_n with the required precision.

Let us emphasize that the integral in (14) is convergent at infinity. The convergence takes place also for other $c_n^{(i)}$. Moreover, this property is a general result of the application of the homogenization [18]. It is worth recalling that the singular terms $1/\tau^2$ and $1/\tau^3$ in (14) are defined in the sense of principal value. Nevertheless, the prescription may be changed (in all above-derived formulas). As a result the coefficients $c_n^{(i)}$ will change, but the asymptotic properties of the expansion (7) will be conserved.

After the corresponding calculations we come to the following result (for the first time obtained in [13]):

$$\begin{aligned} c_0 &= \frac{\pi}{\alpha f_0} + \frac{\pi}{f_0^2}\left(h_0' f_0 - h_0 f_0'\right) + \alpha \frac{\pi}{f_0^3}\Big({h'}_0^2 f_0^2 + h_0^2 {f'}_0^2 \\ &\quad + h_0 h_0'' f_0^2 - 2 h_0 h_0' f_0 f_0' - \tfrac{1}{2} h_0^2 f_0 f_0'' - \tfrac{1}{2} f_0^3 f_0'' \Big), \\ c_1 &= \frac{\pi h_0}{f_0} + \alpha \frac{\pi}{f_0^2}\left(2 h_0 {h'}_0 f_0 - h_0^2 {f'}_0 - f_0^2 {f'}_0\right), \\ c_2 &= \alpha \frac{\pi}{f_0}\left(h_0^2 - f_0^2\right). \end{aligned} \quad (15)$$

Here the subscript 0 means that the corresponding quantities are defined at $\tau = 0$, while the superscript means we are taking derivatives. For example, $h_0' = \mathrm{d}h(\tau)/\mathrm{d}\tau|_{\tau=0}$, etc.

The derived result may be written in a more compact form if the quantities c_n in (9) are understood as functions of τ. In this case, instead of (15), one can obtain

$$c_0 = \frac{\pi}{\alpha f}, \quad c_1 = \frac{\pi h}{f}, \quad c_2 = \alpha \frac{\pi\left(h^2 - f^2\right)}{f}. \quad (16)$$

Here $c_{0,1,2}$, h and f are functions of the τ. The equivalence of these two forms, (15) and (16), follows from the relations $f(\tau)\delta'(\tau) = f_0\delta'(\tau) - f_0'\delta(\tau)$ and $f(\tau)\delta''(\tau) = f_0\delta''(\tau) - 2f_0'\delta'(\tau) + f_0''\delta(\tau)$.

The above-derived results may easily be extended to the case of multi-loop contributions in $h(\tau)$ and $f(\tau)$. In this case one should perform the conventional expansion in α in formulas (15) or (16), in which h and f are understood with the presence of multi-loop contributions [13]. The simplest way to prove this statement is as follows: assuming that h and f are the full functions we repeat all the same reasoning as was done above, except that during homogenization we modify the scaling by setting $\alpha^n \to \xi\alpha^n$ in the n-loop contributions (instead of $\alpha^n \to \xi^n\alpha^n$). As a result the higher-loop contributions become identified with the one-loop ones, and formulas (15) and (16) are restored automatically.

4 The properties of AO expansion of $\mathcal{W}(\alpha;\tau)$

Let us discuss now the properties of the AO expansion of $\mathcal{W}(\alpha;\tau)$. At first we are interested in the characteristic which will be useful in proving gauge cancellations in the probability. It should be emphasized that a solution to this problem is not obvious a priori due to the specificity of the Feynman rules in the modified PT approach (see formulas (7), (9) and (15)). Then we will examine the self-consistency properties of the expansion, such as a non-sensitivity of the results to the ambiguities in the definition of the gauge bosons masses [21], and consistency with the requirements of UV renormalization.

We start from a rather methodological problem consisting in the explicit demonstration of the independence of the results of an AO expansion from the sequence of the expansion in the higher loops. In other words, we show that the result of the AO expansion of $\mathcal{W}(\alpha;\tau)$ will be the same if instead of (1) one will start from the following formula of the incomplete Dyson resummation:

$$\Delta(\alpha;\tau) \equiv \frac{1}{\tau - \alpha\Sigma_1 - \alpha^2\Sigma_2 - \alpha^3\Sigma_3 + \cdots}$$

$$= \frac{1}{\tau - \alpha\Sigma_1} + \frac{\alpha^2\Sigma_2 + \alpha^3\Sigma_3}{(\tau - \alpha\Sigma_1)^2} + \cdots \qquad (17)$$

Here $\alpha^n \Sigma_n(\tau)$ is the n-loop contribution to the self-energy. By squaring (17) we get the incomplete expansion for $\mathcal{W}(\alpha;\tau)$ each term of which is integrable in the conventional sense:

$$\begin{aligned}\mathcal{W}(\alpha;\tau) &= \mathcal{W}_1(\alpha;\tau) + \Big[\left(\alpha^2\Sigma_2 + \alpha^3\Sigma_3\right)\mathcal{W}_{11}(\alpha;\tau) \\ &\quad + \alpha^4\left(\Sigma_2\right)^2 \mathcal{W}_{12}(\alpha;\tau) + \text{h.c.}\Big] + \alpha^4|\Sigma_2|^2\mathcal{W}_1^2(\alpha;\tau) \\ &\quad + O(\alpha^2). \end{aligned} \qquad (18)$$

Here we have introduced three new functions: $\mathcal{W}_{11}(\alpha;\tau) = \mathcal{W}_1(\alpha;\tau)\Delta_1(\alpha;\tau)$, $\mathcal{W}_{12}(\alpha;\tau) = \mathcal{W}_1(\alpha;\tau)\Delta_1^2(\alpha;\tau)$, and $\mathcal{W}_1^2(\alpha;\tau) = \mathcal{W}_1(\alpha;\tau)\mathcal{W}_1(\alpha;\tau)$, where $\mathcal{W}_1(\alpha;\tau)$ and Δ_1 $(\alpha;\tau)$ are the same functions as $\mathcal{W}(\alpha;\tau)$ and $\Delta(\alpha;\tau)$, but with only one-loop self-energies Dyson resummed. By considering these new functions as generalized functions, each term in (18) may be in turn completely AO expanded. Repeating the reasonings of Sect. 2, one can show that the leading term of the AO expansion of $\mathcal{W}_{11}(\alpha;\tau)$ has the behavior of $1/\alpha^2$. So, after integration with the weight function $\varphi(\tau)$ the first term in the square brackets in (18) makes a contribution of order $O(1)$. The leading terms in $\mathcal{W}_{12}$ and $\mathcal{W}_1^2$ both are $O(1/\alpha^3)$. So, the second term in the square brackets and the last term in (18) are of order $O(\alpha)$. By a similar reasoning one can show that the neglected terms in (18) are of the order $O(\alpha^2)$. So in view of $\mathcal{W}_1(\alpha;\tau) = O(\alpha^{-1})$, formula (18) approximates $\mathcal{W}(\alpha;\tau)$ within the NNLO precision.

It should be noted, that the above reasoning is valid only as long as the functions $\Sigma_l(\tau)$, $l = 2, 3, \cdots$, and several of their derivatives are regular functions in some neighborhood of $\tau = 0$. If this is not the case, then the products of $\Sigma_l(\tau)$ on $\mathcal{W}_{1n}^m$ must be considered as new generalized functions, of which the properties should additionally be investigated. Such a situation occurs when the unstable particle interacts with massless particles (photons). This difficulty may be eliminated by the inclusion of the regularizing mass for massless particles, because then functions $\Sigma_l(\tau)$ become regular in the vicinity of $\tau = 0$.

Assuming this trick let us now consider the complete AO expansion of $\mathcal{W}_{1n}^m$. For brevity we omit the corresponding derivation since it is similar to the one considered in the previous section. The desirable accuracy of the expansion is controlled by formula (18), so that $\mathcal{W}_{11}$ should be expanded up to $O(1)$ corrections, and $\mathcal{W}_{12}$ and $\mathcal{W}_1^2$ up to $O(\alpha^{-2})$ corrections. However, in further calculations major precision is required. So, let us consider

$$\begin{aligned}\mathcal{W}_{11}(\alpha;\tau) &= E(\tau) + 1/\tau^3 + O(\alpha), \\ \mathcal{W}_{12}(\alpha;\tau) &= E(\tau) + O(1), \\ \mathcal{W}_1^2(\alpha;\tau) &= E(\tau) + O(1). \end{aligned} \qquad (19)$$

Here, in all three cases $E(\tau)$ is defined by (9), but with different coefficients c_n. In the compact form, when c_n are defined as functions on τ, we get

$\mathcal{W}_{11}$:

$$c_0 = \frac{\mathrm{i}\pi}{2\alpha^2 f^2}, \qquad c_1 = \frac{\pi(\mathrm{i}h + f)}{2\alpha f^2},$$
$$c_2 = \frac{\pi(\mathrm{i}h^2 + \mathrm{i}f^2 + 2hf)}{2f^2}; \qquad (20)$$

$\mathcal{W}_{12}$:

$$c_0 = -\frac{\pi}{4\alpha^3 f^3}, \qquad c_1 = \frac{\pi(\mathrm{i}f - h)}{4\alpha^2 f^3},$$
$$c_2 = \frac{\pi(2\mathrm{i}hf + f^2 - h^2)}{4\alpha f^3}; \qquad (21)$$

$\mathcal{W}_1^2$:

$$c_0 = \frac{\pi}{2\alpha^3 f^3},$$
$$c_1 = \frac{\pi h}{2\alpha^2 f^3}, \qquad c_2 = \frac{\pi(h^2 + f^2)}{2\alpha f^3}. \qquad (22)$$

Notice here that the coefficients singular in α do not depend on the prescription for the poles in τ, since in the examples considered above the non-integrable terms (for which the prescription is needed) are non-singular in α.

Based on the derived results we formulate the following properties of the expansion. (All of them can be verified by direct complete expanding and comparing the results.)

Property 1. The incomplete expansion (18) is equivalent within the given precision to the complete AO expansion of $\mathcal{W}(\alpha;\tau)$.

The next properties are non-trivial and some additional argument should be given.

Property 2. Within the given precision an incomplete expansion of $\mathcal{W}(\alpha;\tau)$ is equivalent to its complete AO expansion if this incomplete expansion includes as Dyson resummed all contributions non-zero at $\tau = 0$ to $\mathrm{Im}\Sigma_1(\tau)$. All other contributions to $\Sigma_1(\tau)$ may be transferred from denominators to numerators in the sense of a conventional expansion. Moreover, only a finite number of terms of the latter expansion are relevant within the given precision (see the remark at the end of the proof).

We perform the proof in two steps. At first we will show that all contributions to $\Sigma_1(\tau)$ zero at $\tau = 0$ without loss of precision may be expanded in the conventional sense. Then we will show that the same operation may be done also for the whole of the real part of $\Sigma_1(\tau)$.

So let $\Sigma_1(\tau) = \Sigma_{01}(\tau) + \widetilde{\Sigma}_1(\tau)$, where by definition $\widetilde{\Sigma}_1(0) = 0$, but $\Sigma_{01}(0) \neq 0$. Since in some neighborhood of $\tau = 0$ $\widetilde{\Sigma}_1(\tau)$ is a correction to $\Sigma_{01}(\tau)$, its contribution may be expanded like $\Sigma_n(\tau)$ with $n > 1$ in (18):

$$\begin{aligned}&\mathcal{W}(\alpha;\tau) \\ &= \mathcal{W}_1(\alpha;\tau) + \Big[\Big(\alpha\widetilde{\Sigma}_1 + \alpha^2\Sigma_2 + \alpha^3\Sigma_3\Big)\mathcal{W}_{11}(\alpha;\tau) \\ &\quad + \Big(\alpha^2\widetilde{\Sigma}_1^2 + 2\alpha^3\widetilde{\Sigma}_1\Sigma_2 + \alpha^4\Sigma_2^2\Big)\mathcal{W}_{12}(\alpha;\tau) + \text{h.c.}\Big] \\ &\quad + \Big[\alpha^2|\widetilde{\Sigma}_1|^2 + \alpha^3\Big(\widetilde{\Sigma}_1 \overset{*}{\Sigma}_2 + \text{h.c.}\Big) + \alpha^4|\Sigma_2|^2\Big]\mathcal{W}_1^2(\alpha;\tau) \\ &\quad + O(\alpha^2). \end{aligned} \qquad (23)$$

Here the symbol "$*$" means complex conjugation. In the denominators of $\mathcal{W}_{1n}^m$ only the $\Sigma_{01}(\tau)$ is Dyson resummed. The remainder in (23) is estimated in the AO sense. The AO expansions of $\mathcal{W}_1$, $\mathcal{W}_{11}$, $\mathcal{W}_{12}$ and $\mathcal{W}_1^2$ within the required precision were previously described in this section.

The proof of (23) may be given noticing that within the given precision the quantity $\alpha\widetilde{\Sigma}_1$ gives a non-zero contribution only if it is raised to not more than the second power. In fact, against the background of the regular terms this property is obvious. If $\widetilde{\Sigma}_1^2$ (or $|\widetilde{\Sigma}_1|^2$) is multiplied by a counterterm originating from $\mathcal{W}_{1n}^m$, the result is non-zero only in the presence of second- or higher-order derivatives of the delta-function in the counterterm (otherwise there acts the property $\widetilde{\Sigma}_1(0) = 0$). Consider the functions $\mathcal{W}_{1n}^m = [\mathcal{W}_1]^m \Delta_1^n$, $n \geq 0$, $m \geq 0$, $n+m-1$ is the number of self-energy insertions on one side of the cut of the diagram of unitarity, $m-1$ is the number on the other side. In these functions such counterterms appear in order $\alpha^{-(n+2m-1)} \times \alpha^2$, and only those functions $\mathcal{W}_{1n}^m$ are to be multiplied by $\widetilde{\Sigma}_1^2$ (or $|\widetilde{\Sigma}_1|^2$) which satisfy the condition $n+2m-2 \geq 2$. Note that in both insertions of $\widetilde{\Sigma}_1$, each gives a factor α. Of the other remaining $n+2m-4$ insertions of $\widetilde{\Sigma}_k$, $k \geq 2$, each gives a factor not less than α^2. Thus all considered terms result in a $O(\alpha)$ contribution. While extending the above reasoning to the third and the higher powers of $\alpha\widetilde{\Sigma}_1$, we easily see that they give non-zero contributions of $O(\alpha^2)$ only.

The above result may be generalized to the real part of $\widetilde{\Sigma}_1(\tau)$. In this case $\widetilde{\Sigma}_1(\tau)$ in (23) is to be defined by $\mathrm{Im}\widetilde{\Sigma}_1(0) = 0$ with $\mathrm{Im}\Sigma_{01}(0) \neq 0$. The basis for this generalization is the observation that the real part of the self-energy does not contribute to the leading-order term of the AO expansion of $\mathcal{W}(\alpha;\tau)$. Let us remark that at first sight formula (23) with this modification should look much more complicated, because from a formal point of view it should contain an infinite series of terms $[\alpha\mathrm{Re}\Sigma_1(\tau)]^n \times \mathcal{W}_{1n}$, each of which is of order $O(\alpha^{-1})$, etc. However, forming the groups with other functions $\mathcal{W}_{1n}^m$ all superfluous terms must mutually cancel. The mentioned groups will be formed by virtue of relations of the type $2\mathrm{Re}\mathcal{W}_{12} + \mathcal{W}_1^2 = O(\alpha^{-1})$ [not $O(\alpha^{-3})$], etc.

It is worth noticing that the above result permits one to expand the bosonic corrections to the self-energy of W- and Z-bosons within a *finite* number of terms. (This is because the mentioned bosonic corrections possess the property $\mathrm{Im}\Sigma_1^{\mathrm{bos}}(0) = 0$ [5,22].) This is a non-trivial result since according to formal power counting all these terms are of the same order in α and, consequently, should be explicitly taken into consideration.

Property 3. There is the following approximation of $\mathcal{W}(\alpha;\tau)$ up to $O(\alpha^n)$ corrections:

$$\mathcal{W}(\alpha;\tau) = \mathcal{W}_{[n]}(\alpha;\tau) - \alpha^{n-1}\frac{\mathrm{Im}\Sigma_{n+1}(0)}{[\mathrm{Im}\Sigma_1(0)]^2}\,\pi\delta(\tau) + O(\alpha^n). \tag{24}$$

Here $\mathcal{W}_{[n]}(\alpha;\tau)$ stands for the propagator squared with Dyson resummed contributions to the self-energy up to n loops. The second term includes the $(n+1)$-loop correction which is necessary (with the appropriate factors) within the indicated precision. Formula (24) follows immediately from the obvious generalization of (18) to the case of any n taking into consideration the first result in (20).

It is worth noticing that the second term in (24) makes a contribution in the highest order within the given precision. Consequently, any $O(\alpha)$-variation of the mass is worthless in this term, because the effect may be referred to the neglected term of order $O(\alpha^n)$. Actually this is a very important observation, since in the case of electroweak theory it permits one to leave out of account the ambiguities in the definition of the gauge boson masses [21]. In particular, one may consider the factor $\mathrm{Im}\Sigma_{n+1}(0)$ to be defined at the complex-value pole position in the spirit of the pole scheme (as a result $\mathrm{Im}\Sigma_{n+1}(0)$ becomes completely gauge invariant), or simply as the n-loop correction to the width of the unstable particle divided by its mass.

Formula (24) represents the quantitative characteristic of the property of the loss of one-loop PT order in the resonance region. Indeed, as long as $\mathcal{W}_{[n]}(\alpha;\tau) \sim \alpha^{-1}$ at $\alpha \to 0$, the second term in (24), represents the n-th order correction, but not the $(n+1)$-th one, as might naively be expected. Since the second term in (24) cannot be obtained from the analysis of an amplitude, it is pertinent to call it the *anomalous additive term.*

Property 4. The above expansions possess the following transformation property when the argument τ is shifted by a quantity of order $O(\alpha)$:

$$\widetilde{\mathcal{W}}(\alpha;\tau) = \widetilde{\mathcal{W}}(\alpha;\tau-\alpha m^2)\Big|_{\substack{h(\tau-\alpha m^2)\to h(\tau)-m^2\\ f(\tau-\alpha m^2)\to f(\tau)}}. \tag{25}$$

Here $\widetilde{\mathcal{W}}(\alpha;\tau)$ stands for any incomplete or complete AO expansion of the considered above functions; the quantity m^2 is of order $O(1)$.

Property (25) indicates the non-sensitivity of the entire formalism with respect to variation of the mass shell within $O(\alpha)$. Moreover, it means independence of the formalism from the UV renormalization scheme. Indeed, at the one-loop level the transition, for example, from the MS scheme to the on-mass-shell (OMS) scheme is described by[5]

$$\begin{aligned}\Sigma_{\mathrm{OMS}}(p^2) &= \Sigma(p^2) - \mathrm{Re}\Sigma(M_{\mathrm{OMS}}^2)\\ &\quad - (p^2 - M_{\mathrm{OMS}}^2)\mathrm{Re}\Sigma'(M_{\mathrm{OMS}}^2),\\ M_{\mathrm{OMS}}^2 &= M^2 - \mathrm{Re}\Sigma(M_{\mathrm{OMS}}^2).\end{aligned} \tag{26}$$

It is obvious that (26) belongs to the class of transformations covered by (25). The transformation at the multi-loop level is controlled by formula (18). So this can be done in accordance with the standard recipes of the UV renormalization, which do not depend on the presence of the

[5] Remember, we do not consider contributions to the numerators of propagators. Therefore, we disregard the multiplicative wave function renormalization

"infrared" counterterms located on the mass shell (see also [17] and references therein). Let us note that the transformation to another scheme of UV renormalization may be done (speculatively) before the squaring of the amplitude and the AO expansion. The consequent AO expansion by no means "feels" in what scheme the Green functions have been renormalized.

The property (25) is trivial in cases of non-expanded $\widetilde{\mathcal{W}}$ or its formal expansions. The non-trivial aspect is that it remains valid for the counterterms $E(\tau)$, too. However, this also can be understood if one notes that the transformation $\tau \to \tau - \alpha m^2$ does not affect the structure of the homogenization at the scaling $\tau \to \xi\tau$, $\alpha \to \xi\alpha$ (see Sect. 3). As a result, the property (25) is valid in the most general case.

Finally, we present one more property which represents doubtless independent interest.

Property 5. From (25) follows the next recurrent formula for the coefficients c_n:

$$c_{n-1} = \frac{1}{n}\left[\frac{1}{\alpha}\frac{\partial}{\partial h_0} - \sum_{r=0}^{N-n}\left(h_0^{(r+1)}\frac{\partial}{\partial h_0^{(r)}} + f_0^{(r+1)}\frac{\partial}{\partial f_0^{(r)}}\right) + \frac{\partial}{\partial M^2}\right] c_n. \qquad (27)$$

Here $c_n = c_n(M^2;\alpha;h_0,\ldots h_0^{(N-n)};f_0,\ldots f_0^{(N-n)})$ are the constants independent from τ, n runs over values $0 \le n \le N$, where N is the maximum degree of the derivative of the delta-functions in $E(\tau)$. Formula (27) is written down with a possible dependence of the coefficients c_n on the parameter M^2. An essential point for the derivation of (27) is the expansion in powers of α of the delta-function $\delta(\tau - \alpha m^2)$ and its derivatives in the r.h.s. of (25).

The practical value of (27) consists in the opportunity to check the results of calculation of counterterms, or to determine the "lower" coefficient c_{n-1} to within $O(\alpha^L)$ if the "higher" coefficient c_n is known to within $O(\alpha^{L+1})$.

5 Massless particles exchange

The problem of taking into consideration massless particles (photons) requires a special analysis because massless-particle contributions to the self-energy of a massive (unstable) particle involve the singularity of $\tau \times \ln(\tau - \mathrm{i}0)$. The first derivative of this expression is not defined at zero. So already the first corrections to the coefficients (15) become uncertain.

One way to solve this problem is to introduce a regularizing mass for massless particles (a soft-photon mass). Then the non-analyticity at $\tau = 0$ disappears, and this fact opens a way to use without problems the above-derived formulas of the AO expansions. In a properly defined probability, taking into account radiation of the real soft photons, all contributions singular in the photon mass should cancel among themselves. That leads to continuity of probability with respect to the photon mass. So, while solving qualitative problems, one could not worry about the presence of the photon mass, because in the final results the dependence on it may be eliminated by the ordinary passage to the zero-mass limit.

It should be stressed that in the above discussion the photon mass should not necessarily be infinitesimal. Moreover, it may be chosen to be so large as is needed for the solution of a particular qualitative problem. In what follows we use the photon-mass insertion for the consistent determination of the orders of expansion in the coupling constant of the photon contributions to the probability. After completion of all cancellations and after the passage to the zero-mass limit these orders will not change. It is worth noticing that the photon mass may be introduced in such a way as to guarantee validity of WI. This can be seen by consideration of the problem in the Stueckelberg formalism[6]. Moreover, a photon mass (and a gluon mass, as well) may be introduced in a totally gauge-invariant fashion [23]. So, the photon mass will not break down gauge cancellations.

Another way [13] to solve the problem of massless particles is based on usage of the regularization property of the parameter α. This method is deeply involved in the context of the AO expansion. It should be noted, however, that it is able to cure only those IR divergences of which the origin is connected with the emergence of an additional singularity in α at $\alpha \to 0$. In reality such divergences appear only in the diagrams where the soft momenta of massless particles come into the propagator of unstable particle considered on shell. Let us note that cancellation of these IR divergences means cancellation of the corresponding singularities in the coupling constant, and vice versa. The idea of the method of [13] consists of a stepwise expansion of the full Green functions squared: first in the contributions of the massless particles only, and then in other vertices using the AO technique. (The first step will not lead to an infinite series of equal orders in α in the resonance region owing to Property 2 and the property $\mathrm{Im}\Sigma_1^{\mathrm{bos}}(0) = 0$ for the massless-particle contributions.)

Acting in the framework of the method of [13], after the first step one gets the modified Green functions with unstable propagators not containing contributions of the massless particles. However, this simplification is not indisputable, since there will appear a lot of configurations with explicit contributions of photons for which some special counterterms will be needed. (In fact they will regularize the products $\Sigma_l \times \mathcal{W}_{1n}^m$; see the discussion in the previous section.) Basing ourselves on considerations of unitarity one can justify cancellation of the considering class of IR divergences [13]. However, the property of the gauge cancellations still remains to be proved. So this method of handling the IR divergences seems not too good, especially since it does not provide a complete solution of the problem and requires a lot of additional efforts.

[6] The author is grateful to A.A. Slavnov for indication of this fact

6 The gauge cancellations

Now we are ready to give a proof of the gauge cancellations in any order of the modified PT. We present our argumentation in a rather general form, applied basically to any unstable particle in electroweak theory. The main idea is to separate in the probability the contributions certainly possessing the property of gauge cancellations from the problematic contributions. That allows us to concentrate on the study of the problematic contributions only.

Let us begin with some preliminary notes. First of all we determine the photon-mass regularization for IR divergences. The advantages of this method have been discussed in the previous section. Here we stress that this method permits one to consider soft photons like ordinary massive particles. That means that their contributions either may be referred completely to vertex blocks, or, in other cases, they suppress the resonant behavior of the corresponding contributions. Really, an exchange by a massive particle suppresses the *on-shell* propagating of unstable particles both before and after the emission of the massive particle. As a result, the problem of instability is reduced solely to configurations which include only the pairs of unstable propagators of equal mass and momenta, placed on both sides of the cut of unitarity. It is worth remembering that the mass insertion for photons will not violate gauge cancellations (see previous section). Moreover, since the probability is a continuous function of the photon mass the dependence on it may be eliminated in the final results by the ordinary passage to the zero-mass limit.

The next note concerns the unphysical pole contribution to the vector boson propagators. Assuming the parameterization for the self-energy

$$\Sigma_{\mu\nu}(p) = \Sigma g_{\mu\nu} + \Sigma_L p_\mu p_\nu, \tag{28}$$

we come to the following explicit expression for the full propagator in R_ξ-gauge:

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2 - M^2 + \Sigma} + \xi \frac{p_\mu p_\nu/p^2}{p^2 - \xi M^2 + \xi\left(\Sigma + p^2 \Sigma_L\right)}. \tag{29}$$

Here the first term represents the product of the spin factor on the scalar propagator $\Delta(\alpha;\tau)$ introduced in (1). The second term describes the unphysical contribution. In the conventional PT it is cancelled due to WI by contributions of the other unphysical states. Consequently, the second term in (29) does not lead to a non-integrable singularity in the phase space. In the modified PT this fact allows us to take into account the second term in the conventionally expanded form. Then the property of gauge cancellations will be controlled by the properties of the first term, taken in an absolute value and having been squared. Note that the cross terms resulting from the squaring of the propagator (29) will not lead to non-integrable singularities in the phase space. So they also may be taken into account as conventionally expanded.

Now let us show that the first term in (29), being taken into consideration in framework of the modified PT, does not break the gauge cancellations in the probability. Remember that in accordance with (24) its contribution may be presented in the form of a sum of two terms, where one is conventional and the other is anomalous. Let us start from an analysis of the anomalous term contribution. Due to its additivity the simplest way to perform the analysis is to assume for a moment that the squared unstable propagator consists in this anomalous term only.

Firstly we consider the case with a single unstable particle production. Let us note that the complementary part of diagram of unitarity with respect to the given propagator squared, by virtue of the delta-function *without a derivative* in the anomalous term, is taken strictly on mass shell. Moreover, the details [21] of the definition of the mass shell are inessential here and may be omitted. (This is because these details have the meaning of corrections, but the anomalous term describes the highest order within the given precision.) From these two facts follows the conclusion that the sum of the complementary parts is gauge invariant, since actually it coincides (up to factors) with the product of two S-matrix elements squared, with one describing the on-shell production of the unstable particle and the other one describing its decay. Furthermore, in the anomalous term itself one may neglect the gauge-dependent contributions, if in this case they are in $\mathrm{Im}\Sigma_{n+1}(0)$, since they are beyond the given precision (see Property 3 and the discussion therein). So the total sum of unitarity diagrams that include the anomalous term within the given precision is gauge invariant.

In the case of multiple production of unstable particles let us single out any unstable particle and, again, consider its contribution of an anomalous term. Since the anomalous term is of the highest order, the contributions of all other unstable particles should be considered in the leading order of the AO expansion. By virtue of (7), (9), and (15) the leading-order contributions are determined by the delta-function *without a derivative* with a coefficient including the one-loop f_0 only as a non-trivial factor. In view of the gauge invariance of the one-loop f_0 (see, for instance, [5]) follows, once again, the gauge invariance of the total sum of complementary parts of the diagrams of unitarity. So, the gauge invariance in this case occurs, too. Due to additivity, a similar reasoning may be repeated for any other unstable particle of multiple production, and in each case we will reach the property of gauge invariance of the corresponding contribution to the probability.

Now let us proceed to the analysis of all other contributions, which do not at all include an anomalous term. Such contributions are controlled by the first term in (24). Also, all non-resonant contributions belong to this class of contributions. As a whole, these contributions are described by the amplitude squared determined in the conventional approach with Dyson resummation. The property of gauge cancellations in the amplitude so determined has already been shown [10] in the framework of the background-field formalism. Let us emphasize that the only condition which has to be observed when considering these contributions is the Dyson resummation of all corrections to the self-energies up to n loops together with taking into consideration all n-loop corrections to the vertices. So, treating $\mathcal{W}_{[n]}(\alpha;\tau)$ in (24) in such a manner, we automatically get the property of gauge cancellations in the probability[7].

The above discussion completes a construction which provides both the gauge cancellations and the necessary precision of the description in the sense of an expansion in the coupling constant. Basing ourselves on this result one may do the next step: proceeding to the complete AO expansion of the probability. Then the property of gauge cancellations within the given precision must take place as well, because AO only expands in powers of the coupling constant the expression which possesses the property of gauge cancellations. With this note we complete the general proof of gauge cancellations in the approach of the modified PT.

7 Generalization of the fermion-loop scheme

If the analysis may be restricted by NLO precision, for example for the case of W-pair production studied at LEP2, the gauge cancellations may be proved within the usual formalism, without applying the background-field method. The key point is the well-known result on gauge cancellations in the so-called fermion-loop scheme [8,9]. Recall that it consists of including all fermionic one-loop corrections in tree-level amplitudes and Dyson resumming self-energies. The difficulty of this scheme is twofold. First, it is not known how to incorporate the one-loop bosonic corrections into this scheme without spoiling the gauge invariance. Second, there is a problem with gauge invariance while taking into account the two-loop corrections to self-energies in the denominators of unstable propagators, which is necessary in the resonance region.

Both these problems may be solved within the modified PT approach with usage of the AO technique. Let us begin with the problem of the two-loop corrections. As we have seen above, they can be taken into consideration without breaking gauge invariance by adding the anomalous terms to the probability. In view of (24) that may be done by means of the formula

$$\mathcal{W}(\alpha;\tau) = \mathcal{W}_{[1]}(\alpha;\tau) - \frac{\mathrm{Im}\Sigma_2(0)}{[\mathrm{Im}\Sigma_1(0)]^2}\,\pi\delta(\tau) + O(\alpha). \quad (30)$$

Here $\mathcal{W}_{[1]}(\alpha;\tau)$ represents the unstable propagator squared with Dyson resummed all one-loop corrections. Remember that $\mathcal{W}_{[1]}(\alpha;\tau) = O(\alpha^{-1})$ at $\alpha \to 0$. Therefore, the second term in (30) describes the highest-order (NLO) correction. So, any ambiguities in its definition, in particular those which concern the problem of the gauge invariance, may be referred to the discarded terms of order $O(\alpha)$. (See Property 3 and the discussion therein.)

The problem of one-loop bosonic corrections may be solved, too. Let us group these corrections into two classes. To the first class we refer the corrections to self-energy of unstable particles. To the second class we refer the corrections to the vertex factors, the corrections due to exchange processes, and due to the real (soft) photons.

The corrections of the first class can easily be taken into the consideration by means of the fact that the imaginary parts of the on-shell bosonic corrections to the self-energies of W- and Z-bosons are zero [5,22]. Owing to this fact and Property 2 of Sect. 4 these corrections may be transferred from the denominators of unstable propagators to their numerators without loss of precision. Moreover, in an OMS-like scheme of UV renormalization, where the renormalized self-energies satisfy the condition $\mathrm{Re}\Sigma_1(0) = \mathrm{Re}\Sigma_1'(0) = 0$, one has the following approximation:

$$\mathcal{W}_{[1]}(\alpha;\tau) = \mathcal{W}_{1F}(\alpha;\tau) + O(\alpha). \quad (31)$$

Here $\mathcal{W}_{1F}(\alpha;\tau)$ represents the propagator squared where only the fermionic one-loop corrections are Dyson resummed. Substituting (31) into (30) we come to the formula which leads to gauge cancellations. In order to see this, we should only repeat the reasoning of the previous section keeping in mind the gauge invariance in the fermion-loop scheme.

However, the bosonic corrections of the second class have not yet been taken into account. In order to do that let us make use of the fact that within the NLO approximation and in the presence of the *corrections*, the quantity $\mathcal{W}_{1F}(\alpha;\tau)$ must be taken into consideration in the leading-order approximation only. Remember that in this approximation $\mathcal{W}_{1F}(\alpha;\tau) = \pi/(\alpha f_{0F}) \times \delta(\tau)$, and this expression is explicitly gauge invariant. Moreover, since the delta-function enters without derivatives, the sum of all factors appearing in its presence in the unitarity diagrams are also gauge invariant (see previous section).

Thus, we come to the following recipe of the generalization. We formulate it having in mind the total cross-section for the typical LEP2 processes CC10, CC11 and CC20, which have been studied in the framework of the fermion-loop scheme [9]. In fact, the generalization consists of adding to the probability two types of corrections.

The corrections of the first type describe the anomalous contributions. In the NLO approximation they look like the cross-section of the pair on-shell production of unstable particles taken in the leading-order approximation, times the leading-order decay blocks of unstable particles, and times the "anomalous" factor. Indeed, the presence of the anomalous factor means that all other factors should be taken in the leading order. However, in the leading-order approximation only the double-resonant subprocesses contribute to $e^+e^- \to 4f$. (In fact these subprocesses are of CC03 class.) That follows from the fact that only these subprocesses include the factor $1/\alpha^2$ originating from the product of two unstable propagators squared. Now, let us remember that the additive anomalous term in (30) includes the delta-function as well as

[7] It should be noted that within the background-field formalism after all gauge cancellations there may remain some residual dependence on the quantum gauge parameter. This phenomenon has been considered in the original work of [10], where it has been stressed also that this residual dependence will not affect the high-energy behavior of the amplitude. An idea for how to solve the problem of the residual gauge dependence is discussed in Sect. 8

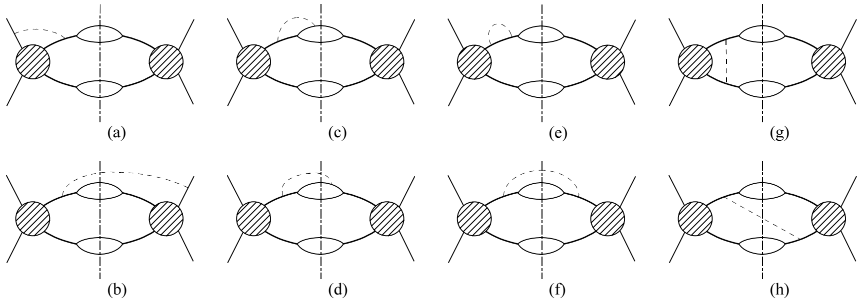

Fig. 1a–h. Examples for factorizable and (quasi-) non-factorizable corrections to W-pair production in $e^+e^- \to 4f(\gamma)$ which have to be taken into consideration in the NLO approximation. (The dashed lines denote massive bosons or soft photons. The continuous thin and thick lines represent the initial/final fermions and the unstable W-bosons, respectively. The vertical dot-dashed lines indicate the cut of the diagrams of unitarity. The hatched areas denote the lowest-order Green functions for the production of the virtual W-boson pair)

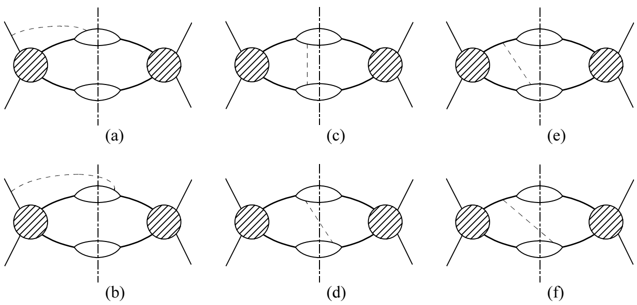

Fig. 2a–f. Examples for non-factorizable corrections to W-pair production which contribute beyond the NLO approximation

$\mathcal{W}_{1F}(\alpha;\tau)$ does in the leading-order approximation. So, the anomalous terms always contribute "on shell".

The corrections of the second type may be represented, again, as the cross-section of the pair on-shell production of unstable particles, multiplied by their decay blocks. However, instead of "anomalous" factors, they include the bosonic corrections to the vertex blocks and the corrections due to the real soft photons. The examples of the unitarity diagrams which generate contributions of this type are shown in Fig. 1. With the non-zero mass of the photons all these diagrams include exactly *two* pairs of unstable propagators of identical mass and momenta in both sides of the cut. (The only exception, configuration (1.e), is discussed below.) Therefore, they include *two* delta-functions of the corresponding kinematic variables, which make these configurations on shell. The sum of all such configurations represents the product of the cross-section of the pair on-shell production of unstable particles and their decay blocks. Since these quantities are continuous functions of the photon mass, the dependence on it may be eliminated in the very end of the calculation by the ordinary limiting procedure. Notice that the mentioned property of continuity follows directly from the well-known theorem for cross-sections with the real soft-photon contributions.

Now let us discuss the above-mentioned exceptional configuration (1.e). Strictly speaking, it should not be considered to belong to the unitarity diagrams, since it describes the self-energy correction to the unstable propagator, which has already been taken into consideration by formula (31). Nevertheless, while considering the cross-section of the pair on-shell production, one has to take into account the virtual soft-photon insertion to the external legs, which is due to the wave function renormalization. The configuration (1.e) was added to the list of diagrams of Fig. 1 only in order to indicate this fact.

It should be noted that among the diagrams of Fig. 1 there are both factorizable and non-factorizable configurations in the sense of the classification of [7, 11, 12]. Nevertheless, at the intermediate step, when the soft photons are supplied with the mass, the non-factorizable configurations of Fig. 1 become factorizable. All other non-factorizable corrections, which do not provide this property, produce such configurations that include *less* than two pairs of unstable propagators of identical mass and momenta in both sides of the cut. Therefore, they do not include the leading-order factor $1/\alpha^2$. In view of the presence of the additional factor α due to the bosonic corrections they contribute beyond the NLO approximation. The examples of configurations of this type are shown in Fig. 2. It is worth noticing that they may not be considered as corrections to the cross-section discussed above of the pair on-shell production of unstable particles, or to their decay blocks.

The above discussion leads us to the following formulas for the total cross-section:

$$\sigma(s) = \int_0^1 \mathrm{d}z\, \phi(z;s)\, \hat\sigma(zs), \tag{32}$$

$$\hat\sigma(s) = \int_0^s \mathrm{d}s_+ \int_0^{(\sqrt{s}-\sqrt{s_+})^2} \mathrm{d}s_-\, \hat\sigma_0(s;s_+,s_-). \tag{33}$$

Here $\sigma(s)$ is the experimentally measured cross-section at the center-of-mass energy squared s. $\phi(z;s)$ is the "flux" function[8] describing the contributions of the initial- and

[8] In the general case, $\phi(z;s)$ includes experimental cuts and hardware factors

final-state photon radiations with large IR and collinear logarithms; $\hat{\sigma}(s)$ stands for the hard scattering improved Born cross-section at the reduced center-of-mass energy squared (see [1,4] and references therein). It should be emphasized that $\hat{\sigma}(s)$ contributes like a distribution to $\sigma(s)$, since $\hat{\sigma}(s)$ is smeared by the flux function over a wide range of the kinematic variable ($\phi(z;s)$ is peaked at $z=1$ and has a tail till a cut at lower values of z). Expression (33) represents $\hat{\sigma}(s)$ in a form with explicit integration over the virtualities of unstable particles (over the invariant masses of the corresponding final states). The quantity $\hat{\sigma}_0(s)$ reads

$$\begin{aligned}
\hat{\sigma}_0(s;s_+,s_-) &= \hat{\sigma}_0^{\text{fermion-loop-scheme}}(s;s_+,s_-)\\
&-\hat{\sigma}_0^{\text{on-shell,tree}}(s;M_+,M_-)\times 2\alpha\mathrm{Im}\Sigma_2(0)/\mathrm{Im}\Sigma_1(0)\\
&\times\prod_{\kappa=\pm}\delta(s_\kappa-M_\kappa^2)\times\mathrm{BR}_\kappa^{\text{tree}}\\
&+\hat{\sigma}_0^{\text{on-shell,tree}}(s;M_+,M_-)\\
&\times\prod_{\kappa=\pm}\delta(s_\kappa-M_\kappa^2)\times\mathrm{BR}_\kappa^{\text{tree/boson-one-loop+real-photon}}\\
&+\hat{\sigma}_0^{\text{on-shell, boson-one-loop + real-photon}}(s;M_+,M_-)\\
&\times\prod_{\kappa=\pm}\delta(s_\kappa-M_\kappa^2)\times\mathrm{BR}_\kappa^{\text{tree}}. \qquad (34)
\end{aligned}$$

Here the first term represents the result of the conventional fermion-loop scheme. All other terms describe its generalization. The factor 2 in the second term reflects the presence of two intermediate unstable particles of equal masses; $\mathrm{BR}_\pm^{\text{tree}}$ denotes their on-shell branching in the tree approximation. In the third term one BR is determined with the bosonic corrections to the partial width. (In fact the third term includes the sum of two subterms, with the modified BR for one of two unstable particle.) In (34) we have used the relation $\alpha\mathrm{Im}\Sigma_1(0)=M\Gamma_0(M)$, with $\Gamma_0(M)$ being the total on-shell width at tree level. This relation follows from unitarity and may be verified by direct calculation [22].

Formula (34), in principle, may be further simplified by carrying out the complete AO expansion in its first term. The key formula for this simplification, written in an OMS-like scheme of UV renormalization, is as follows:

$$\mathcal{W}_1(\alpha;\tau)=[\alpha\,\mathrm{Im}\Sigma_1(0)]^{-1}\;\pi\delta(\tau)+VP\frac{1}{\tau^2}+O(\alpha). \qquad (35)$$

Substituting (35) into (34) we finally get

$$\begin{aligned}
\hat{\sigma}_0(s;s_+,s_-) &= VP\;\hat{\sigma}_0^{\text{on/off-shell, tree}}(s;M_+,s_-)\\
&\times\delta(s_+-M_+^2)\times\mathrm{BR}_+^{\text{tree}}\\
&+VP\;\hat{\sigma}_0^{\text{off/on-shell, tree}}(s;s_+,M_-)\times\delta(s_--M_-^2)\\
&\times\mathrm{BR}_-^{\text{tree}}\\
&-\hat{\sigma}_0^{\text{on-shell,tree}}(s;M_+,M_-)\times 2\alpha\mathrm{Im}\Sigma_2(0)/\mathrm{Im}\Sigma_1(0)\\
&\times\prod_{\kappa=\pm}\delta(s_\kappa-M_\kappa^2)\times\mathrm{BR}_\kappa^{\text{tree}}\\
&+\hat{\sigma}_0^{\text{on-shell,tree}}(s;M_+,M_-)\\
&\times\prod_{\kappa=\pm}\delta(s_\kappa-M_\kappa^2)\times\mathrm{BR}_\kappa^{\text{tree/boson-one-loop+real-photon}}\\
&+\hat{\sigma}_0^{\text{on-shell, tree + one-loop + real-photon}}(s;M_+,M_-)\\
&\times\prod_{\kappa=\pm}\delta(s_\kappa-M_\kappa^2)\times\mathrm{BR}_\kappa^{\text{tree}}. \qquad (36)
\end{aligned}$$

Here the first two terms and the leading-order tree contribution in the last term originate from the fermion-loop-scheme term in formula (34). Namely, the first two terms accumulate the contributions originating from CC03 subprocesses and simultaneously from subprocesses of single vector boson production. In the case of CC03 subprocesses the symbol VP means that the corresponding off-shell unstable propagator squared is approximated, in accordance with (35), by $VP1/\tau^2$. The other unstable particle in this case is considered as real, produced on shell. In the case of single production the vector (unstable) boson is considered as real, too. The symbol VP in this case is superfluous and has to be omitted.

It should be noted that in the above discussion the usage of the prescription of the principal value in the case of the CC03 subprocesses is necessary, because the usage of any other prescription for $1/\tau^2$ in (31) and (35) may change the results in (36). This remark, however, does not concern the anomalous term, since it arises from the contribution singular in α to $\mathcal{W}_{11}(\alpha;\tau)$ (see (20) and the note after (22)).

The above results may easily be generalized to other processes with unstable particle production, including the processes of the "NC"-type and of the "mixed" type. In the latter case there appear the additional terms in (34) and (36) of almost the same structure, that describe the ZZ-contributions (of course, the configurations of Fig. 1 are not relevant in the ZZ-case). The generalization for the case of the differential cross-sections should not lead to difficulties either. However, the discussion of this question is beyond the scope of the present paper.

8 Discussion

In this paper we have found a practical method that ensures both the gauge cancellations and a fixed precision of description of processes mediated by the production and decay of unstable particles in electroweak theory. The solution has been found in the framework of the modified PT which consists of an expansion in the coupling constant directly of probabilities instead of amplitudes [13]. In the general case and within any fixed precision the proposed method is based on the results obtained earlier [10] in the framework of the background-field formalism. Within NLO precision and in the case of W- and/or Z-pair production we have found a way also in the usual formalism, basing ourselves on results of the fermion-loop scheme [9]. But in contrast to the fermion-loop scheme we perform the description explicitly taking into account the bosonic corrections and the two-loop corrections to the self-energy of unstable particles in the resonance region. From the practical point of view the latter result apparently is the most important one obtained in this paper.

It should be noted that the result on gauge cancellations in the completely expanded probability generally was expected since the appearance of [13]. This work demonstrated that the calculation of the probability of a process mediated by unstable particle production may be reduced to the regular fixed-order calculation. The latter fact gave a reason to think that the problem of the gauge invariance was solved, too [13]. However, on closer examination of the problem a large distance to obtaining the result was found. Indeed, the Feynman rules in the modified PT and in the conventional PT coincide only out of the mass shell. So, the gauge invariance a priori occurs only in this area of the kinematic variable, but not in the neighborhood of the mass shell, where one has the non-standard contributions of the modified PT. However, due to the delta-functions these non-standard contributions do not vanish irrespectively of that as far as the neighborhood is small. Moreover, the properties of these non-standard contributions are unknown in advance. In particular, it is not known whether they are gauge invariant or not. This becomes especially clear in the higher orders of the AO expansion, beginning with NLO, where the non-standard contributions include the derivatives of the delta-functions.

In the present paper we avoid this difficulty by making use of incomplete AO expansions of the unstable propagators squared. This method allows us to carry out a comparison of the results of the modified PT with those of the conventional PT with Dyson resummation. The special role in this scheme is assigned to the anomalous additive term that corrects the result of incomplete expansions of the propagator squared in the vicinity of the mass shell.

We draw attention to a particular problem that arises when comparing the results obtained in the modified PT with the results obtained in the conventional approach with use of the background-field formalism. (It should be emphasized that there is no such problem in the conventional formalism on using the fermion-loop scheme.) This is the problem of the residual dependence on the quantum gauge parameter which still remains in the background-field formalism after all gauge cancellations [10]. In the general case this dependence may pass on to the results of the modified PT. Nevertheless, *in the probability* owing to the phenomenon of changing of the order of particular contributions, which in turn is a corollary of the singularity in the coupling constant in the propagator squared, one may expect that this dependence will drop out within the given precision. Note that we indicate only an opportunity of giving a solution to the problem, but not the actual solution. However, earlier it was not clear even how to initiate a solution [9,10,12].

Now let us discuss the generalization of the fermion-loop scheme. Remember that it provides both the NLO precision and the gauge invariance. As a matter of fact the generalization consists in the instruction on how to use the already known results of the former calculations. Indeed, the first term in formula (34) and the last two terms are already known [9,24,25]. The remaining second term is actually known too, since $\mathrm{Im}\Sigma_2(0)$ in it may be replaced by a one-loop correction to the width of the W-boson divided by its mass (see Property 3 and the discussion thereof).

The last two terms in (34) present an exhaustive description of the bosonic corrections, which formerly were a “stumbling block” for gauge invariance. In the proposed generalization they are located in the S-matrices squared, where the one S-matrix is for the on-shell production of unstable particles and the other ones are for their decays. The key observation that leads us to this simple result is the absence in the NLO approximation of non-factorizable corrections of Fig. 2 (see Sect. 7). This property demonstrates one of the differences of the approach offered with the well-known double-pole approximation (DPA) [7].

In fact there are more serious differences, too. The most obvious one is as follows. By definition, DPA does not take into account the single-pole contributions, while the fermion-loop scheme, and consequently our generalization, does. The mentioned missed contributions are of order Γ_W/M_W, which amounts to 2–3%. This is too much compared with the current accuracy of LEP2 results. In view of this problem, DPA usually is applied for the calculation of the radiative correction only, but not of the Born term. The Born term is considered in the framework of the conventional PT with “by-hand” substitution of Breit–Wigner's propagators instead of the free propagators for unstable particles [7]. However, the latter operation violates the gauge invariance and leads to some error, too, and a certain estimate of this error does not exist. All that it is possible to say is that this error most likely is somewhat less than the shift of the amplitude caused by this substitution, but it is not clear how far less. Nevertheless, the latter quantity again is $O(\Gamma_W/M_W)$. So, the discussed approximation of the Born term cannot be considered satisfactory.

Another serious difficulty of DPA is its inapplicability in the vicinity of phase-space boundaries. This effect arises as a result of the “mapping” procedure in the phase space or the “analytic continuation” of the constant residues at the double pole in the amplitude [7]. First of all, it manifests itself near the threshold region, where the DPA uncertainties are blowing up. Another evident restriction of the applicability of DPA is the region lying far off from resonance where the pole-scheme expansion cannot be viewed as an effective expansion in powers of Γ_W/M_W. Remember that the proposed generalization of the fermion-loop scheme does not need the mapping procedure and does not use the pole expansion. Consequently, it should be free from these difficulties.

In conclusion, let us discuss two problems that arise in the modified PT approach. In fact, the topic is about a comparison of the results of the modified PT with Breit–Wigner's parameterization of unstable particles. The first problem concerns the definition of the “physical” mass and the width of unstable particles. In this connection it should be noted, first of all, that both these quantities are pseudo-observables that are to be determined on the basis of realistic observables [5,22]. Next, let us note that in the framework of the modified PT both these quantities are secondary ones, which should be determined on the basis of primary objects, such as the renormalized Lagrangian mass, coupling constant, etc. Apparently, the most radical way consists of the identification of both these quantities with the position of the pole in the complex plane of the full unstable propagator (in the spirit of the pole scheme). This procedure is equivalent to finding a solution to the equation $M^2 - s_p - \Sigma(s_p) = 0$ with $s_p = M_p^2 - \mathrm{i}M_p\Gamma_p$. This operation may be done perturbatively [21].

The second problem concerns the presence of the delta-functions and VP's in the results of the complete AO expansion. (The urgency of this problem is not so high while making use of incomplete AO expansions as employed in this paper.) The problem may be designated as an illusory discrepancy between the presence of these singular functions in the amplitude squared and the notion of a continuity of the physical results as functions of the kinematic variables. This problem has been stated and discussed in [13]. The essence of the solution is reduced to the observation that actually there is not a direct identity between the squared amplitude (that is usually calculated) and the genuine probability (that is observed). In fact, a formal expression for the probability, calculated on the basis of the unitarity diagrams, should necessarily be subjected to the operation of *integration*, or "*smearing*" with some weight function before it becomes an observable quantity[9]. In processes with light and charged initial and/or final states such a smearing function is, first of all, the flux function that extracts and exponentiates the infrared and collinear singularities from the cross-section (see formula (32)).

In the case of W- and/or Z-production the smearing is large enough: at some distance from threshold its final effect is about ten or more percent and it covers an area which is certainly greater than the unstable-particle width (see formulas (32) and (33)). Consequently, after the convolution the contributions of the delta-functions and VP's become strongly distributed over a wide area. Mathematically, this fact means that the smearing may be considered as an effect of order $O(1)$, which is required in the AO. Therefore, even in the case of ideal energy resolution in a given experiment, the presence of the singular functions in the results of the AO expansion becomes invisible, at least within the given precision of the description. The above conclusion should hold both for the total and differential cross-sections, since the differential cross-sections are determined through the convolution procedure, too [1].

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[9] It is worth noticing that in the absence of "smearing" there is no problem of non-integrable singularities in the phase space. In the latter case all calculations actually are carried out in the framework of the conventional PT

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